Homework 9 Solutions

1. Let $z, w \in \mathbb{C}$.

   (a) Prove that $\overline{z+w} = \overline{z} + \overline{w}$.

   Let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned}
\overline{z+w} &= \overline{(a+bi)+(c+di)} \\
&= \overline{(a+c)+(b+d)i} \\
&= (a+c)-(b+d)i \\
&= (a-bi)+(c-di) \\
&= \overline{z} + \overline{w}
\end{aligned}$$

   (b) Prove that $\overline{zw} = (\overline{z})(\overline{w})$.

   Let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned}
\overline{zw} &= \overline{(a+bi)(c+di)} \\
&= \overline{(ac-bd)+(ad+bc)i} \\
&= (ac-bd)-(ad+bc)i \\
&= (a-bi)(c-di) \\
&= (\overline{z})(\overline{w})
\end{aligned}$$

2. Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$ are not both zero.

   (a) Show that the eigenvalues of A are $\lambda = a \pm bi$.

   $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2$.

   Setting this equal to zero, we obtain $(a-\lambda)^2 + b = 0$ or $(a-\lambda)^2 = -b^2$. Taking the square root of both sides gives $a - \lambda = \pm\sqrt{-b^2} = \pm bi$. Now solving for λ yields $\lambda = a \pm bi$.

   (b) For the following values of a and b, compute and sketch the vectors

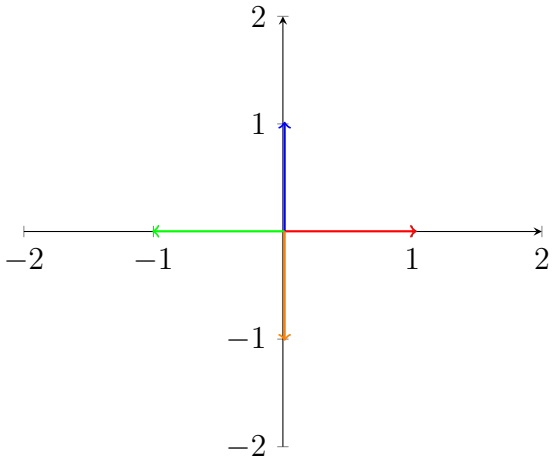
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A^2\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } A^3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

in $\mathbb{R}^2$.

$$(i) \quad a = 0, b = 1 \qquad (ii) \quad a = 1, b = 1 \qquad (iii) \quad a = .6, b = .6$$

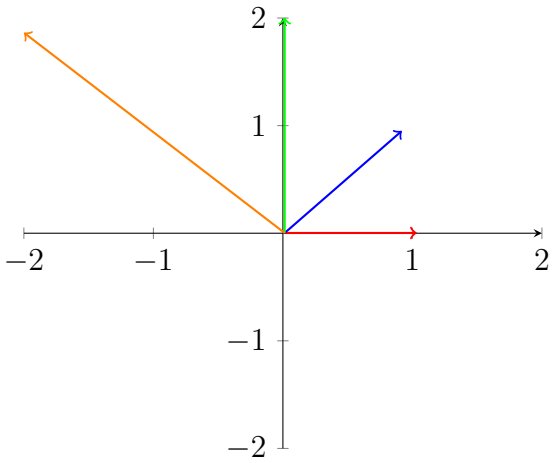
(*i*):

We have $A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$, $A^2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$, and $A^3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$. These vectors are drawn below.



(*ii*):

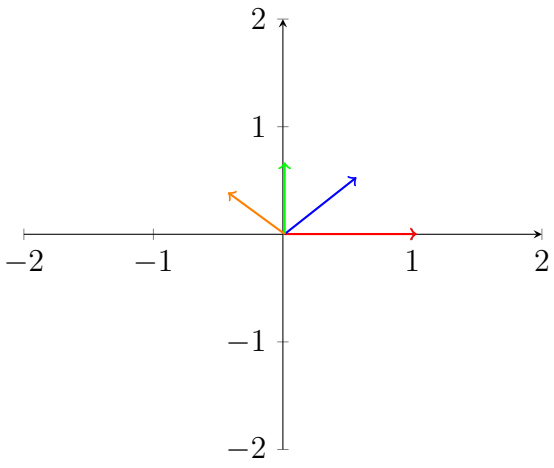
We have $A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$, $A^2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$, and $A^3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$. These vectors are drawn below.



Multiplication by A rotates vectors by 45° and makes them longer (by scaling by a factor of $\sqrt{2}$).

(*iii*):

We have $A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}.6\\.6\end{bmatrix}$, $A^2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\.72\end{bmatrix}$, and $A^3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-.432\\.432\end{bmatrix}$. These vectors are drawn below.



(c) It turns out that multiplication by A rotates and scales vectors. In the cases above, determine the angle and scalar by which A rotates and scales vectors. Explain your reasoning.

(*i*): From the plot drawn above, we see that multiplication by A rotates vectors by $90°$ and scales them by a factor of 1.

(*ii*): From the plot, we see that multiplication by A rotates vectors by $45°$. Since

$$\left\|\begin{bmatrix}1\\0\end{bmatrix}\right\| = 1 \quad \text{and} \quad \left\|A\begin{bmatrix}1\\0\end{bmatrix}\right\| = \left\|\begin{bmatrix}1\\1\end{bmatrix}\right\| = \sqrt{2}$$

multiplication by A scales vectors by a factor of $\sqrt{2}$.

(*iii*): From the plot, we see that multiplication by A rotates vectors by $45°$. Since

$$\left\|\begin{bmatrix}1\\0\end{bmatrix}\right\| = 1 \quad \text{and} \quad \left\|A\begin{bmatrix}1\\0\end{bmatrix}\right\| = \left\|\begin{bmatrix}.6\\.6\end{bmatrix}\right\| = \sqrt{.72}$$

multiplication by A scales vectors by a factor of $\sqrt{.72}$.

(d) Now let's prove that multiplication by A rotates and scales vectors for all such matrices.

(i) Let $r = \sqrt{a^2 + b^2}$. Show that $A = \begin{bmatrix}r & 0\\0 & r\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$ for some θ.

Let $r = \sqrt{a^2+b^2}$. Since a and b are not both zero, $r \neq 0$. Hence Consider a right triangle whose side lengths are $|a|$ and $|b|$; then its hypotenuse has length r. Let θ be the angle between the hypotenuse and the side with length $|a|$. Then using trig identities, we have that $\sin\theta = b/r$ and $\cos\theta = a/r$.
Thus we have
$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(ii) Deduce that multiplication by A rotates and scales vectors.

Recall that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix that rotates matrices by an angle of θ. Thus if $\mathbf{v} \in \mathbb{R}^2$ is any nonzero vector then $A\mathbf{v} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{v}$ is the vector obtained by rotating by an angle of θ and then scaling by a factor of r.

3. Let $S = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) Find the eigenvalues of S.

$$\begin{aligned} \det(S-\lambda I) &= \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} \\ &= -1 \begin{vmatrix} 1 & -1 \\ 2-\lambda & 0 \end{vmatrix} + (2-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \text{ (R3 Cofactor Expansion)} \\ &= -(2-\lambda) + (2-\lambda)((3-\lambda)(2-\lambda)-1) \\ &= -(2-\lambda) + (2-\lambda)(\lambda^2 - 5\lambda + 5) \\ &= (2-\lambda)(\lambda^2 - 5\lambda + 4) \\ &= (2-\lambda)(\lambda-4)(\lambda-1) \end{aligned}$$

Setting this equal to zero, we find the eigenvalues $1, 2, 4$.

(b) Find unit eigenvectors corresponding to each eigenvalue of S

$\underline{\lambda = 1}$: We must solve $(A-I)\mathbf{x} = \mathbf{0}$. Reducing $A - I$ gives

$$A - I = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence x_3 is the free variable and the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. The unit vector in the direction of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

$\underline{\lambda = 2}$: We must solve $(A - 2I)\mathbf{x} = \mathbf{0}$. Reducing $A - I$ gives

$$A - 2I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence x_3 is the free variable and the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

The unit vector in the direction of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

$\underline{\lambda = 4}$: We must solve $(A - 4I)\mathbf{x} = \mathbf{0}$. Reducing $A - I$ gives

$$A - 4I = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence x_3 is the free variable and the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$. The unit vector in the direction of $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ is $\mathbf{u}_3 = \begin{bmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$.

(c) Use the above eigenvalues and eigenvectors to diagonalize S.

By the diagonalization theorem, $S = PDP^{-1}$, where $P = \begin{bmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(d) Write down the statement of the Spectral Theorem and explain why the diagonalization you found in part (c) is an orthogonal diagonalization.

The Spectral Theorem states that every symmetric matrix can be orthogonally diagonalized; that is, $S = QDQ^T$, where Q is an orthogonal matrix of eigenvectors of S and D is diagonal.

Since $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ are unit vectors that belong to distinct eigenspaces of S and eigenspaces of symmetric matrices are orthogonal, these vectors form an orthonomal set. Hence the matrix Q in part (c) is orthogonal (since it is square) and so $P^{-1} = Q$. Thus $S = PDP^T$ is an orthogonal diagonalization.

4. In class we proved that a symmetric matrix S is positive definite if and only if $\mathbf{x}^T S\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$. Use this fact to show that the following matrix is not positive definite.

$$S = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 1 & 1 \\ -1 & 1 & -4 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Since $\mathbf{e}_3^T S\mathbf{e}_3 = -4 < 0$, S is not positive definite.

5. A symmetric matrix is called *negative definite* if all of its eigenvalues are negative. Show that if $\mathbf{x}^T S\mathbf{x} < 0$ for all $\mathbf{x} \neq 0$, then S is negative definite [Note: this can be upgraded to an if and only if statement].

Let λ be an eigenvalue of S and suppose $S\mathbf{v} = \lambda\mathbf{v}$ for some nonzero $\mathbf{v}$. Then

$$0 > \mathbf{v}^T S\mathbf{v} = \mathbf{v}^T(\lambda\mathbf{v}) = \lambda\mathbf{v}^T\mathbf{v} = \lambda\mathbf{v} \cdot \mathbf{v} = \lambda||\mathbf{v}||^2$$

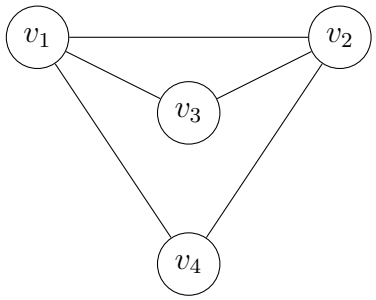
Now since $||\mathbf{v}|| > 0$, we have that $\lambda < 0$.
It follows that all eigenvalues of S are negative and so S is negative definite.

6. In Worksheet 9.26, we defined the incidence matrix of a graph. In this problem, we will consider a different matrix that can be defined from a graph called the **Laplacian matrix**. Let G be a graph with vertices $v_1, \ldots, v_n$. The *degree* of v_i (denoted $\deg(v_i)$) is the number of edges adjancent to v_i. The Laplacian matrix of G is the matrix whose $(i, j)-$th entry is given by

$$\begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ and } v_j \text{ share an edge} \\ 0 & \text{otherwise} \end{cases}$$

It turns out that Laplacian matrices are positive semi-definite. We will see this in an example.

(a) Consider the graph G below. Find the Laplacian matrix of G.



$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

(b) Use Julia to show that L is positive semi-definite. Include your code and output in your solution.

First note that L is symmetric. To show it is positive semi-definite, we find its eigenvalues.

Code:
```
using LinearAlgebra
S=[3 -1 -1 -1; -1 3 -1 -1; -1 -1 2 0; -1 -1 0 2]
valS,vecS=eigen(S)
```

Julia provides the eigenvalues $0, 2$, and 4, which are all nonnegative. Thus L is positive semi-definite.

7. True/False. If a statement is true, prove it. If it is false, explain why or provide a counterexample.

(a) If S is a symmetric $n \times n$ matrix, then $\mathbb{R}^n$ has a orthonormal basis of eigenvectors of S.

True. By the Spectral Theorem, S is orthogonally diagonalizable; that is $S = QDQ^T$, where D is diagonal and Q is an orthogonal matrix of eigenvectors of S. By definition, the columns of Q is an orthormal set of n eigenvectors of S. Since orthonormal vectors are linearly independent, this provides an orthonormal basis for $\mathbb{R}^n$.

(b) Every eigenvalue of a symmetric matrix is nonnegative.

False. $-I$ is symmetric, but evey eigenvalue is -1.

(c) Let A be $m \times n$. If $\lambda \neq 0$ is an eigenvalue of $A^T A$, then it is also an eigenvalue of AA^T.

True. Let $\lambda \neq 0$ be an eigenvalue of $A^T A$; then $A^T A\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \neq 0$. Multiplying by A on the left gives

$$A(A^T A\mathbf{v}) = A(\lambda\mathbf{v})$$
$$(AA^T)A\mathbf{v} = \lambda(A\mathbf{v})$$

Since $\mathbf{v} \neq 0$ (as it is an eigenvector) and $\lambda \neq 0$, we have that $A^T A\mathbf{v} = \lambda\mathbf{v} \neq 0$; it follows that $A\mathbf{v} \neq 0$. Hence λ is an eigenvalue of AA^T with corresponding eigenvector $A\mathbf{v}$.

(d) Let A be $m \times n$. If $\lambda = 0$ is an eigenvalue of $A^T A$, then it is also an eigenvalue of AA^T.

False. Let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has eigenvalues 0 and 1 (the diagonal entries). On the other hand, $AA^T = \begin{bmatrix} 1 \end{bmatrix}$ only has eigenvalue 1.